

Hamiltonian Reduction of Supersymmetric WZNW Models on Bosonic Groups and Superstrings

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Abstract

It is shown that an alternative supersymmetric version of the Liouville equation extracted from D=3 Green–Schwarz superstring equations [1] naturally arises as a super–Toda model obtained from a properly constrained supersymmetric WZNW theory based on the $sl(2, \mathbf{R})$ algebra. Hamiltonian reduction is performed by imposing a nonlinear superfield constraint which turns out to be a mixture of a first– and second–class constraint on supercurrent components. Supersymmetry of the model is realized nonlinearly and is spontaneously broken.

The set of independent current fields which survive the Hamiltonian reduction contains (in the holomorphic sector) one bosonic current of spin 2 (the stress–tensor of the spin 0 Liouville mode) and two fermionic fields of spin $\frac{3}{2}$ and $-\frac{1}{2}$. The $n = 1$ superconformal system thus obtained is of the same kind as one describing noncritical fermionic strings in a universal string theory [2].

The generalization of this procedure allows one to produce from any bosonic Lie algebra super–Toda models and associated super– \mathcal{W} algebras together with their nonstandard realizations.

PACS: 11-17.+y; 11.30.Pb,Qc; 11.40.-q

Keywords: supersymmetry, superstrings, WZNW models, Hamiltonian reduction.

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1 Introduction

In a recent paper [1] a system of constraints and equations of motion describing an $N = 2$, $D = 3$ Green–Schwarz superstring in a doubly supersymmetric (so called twistor–like) formulation [3, 4, 5] was reduced to a superfield system of equations in $n = (1, 1)$, $d = 2$ worldsheet superspace¹, which, in turn, was reduced to the purely bosonic Liouville equation plus two free massless fermion equations. Thus, the supersymmetric Liouville equation was shown to admit another $n = (1, 1)$ supersymmetric realization different from the conventional one [6, 7].

In the present article we show that the same system of equations arises also in a properly constrained $n = (1, 1)$ superconformal Wess–Zumino–Novikov–Witten model [8] with an $Sl(2, \mathbf{R})$ group manifold as the target space.

Supersymmetric WZNW models based on classical Lie groups were considered in detail in [9, 10]. A coset construction of superconformal field theories corresponding to constrained versions of the supersymmetric WZNW models was proposed in [11, 12] (for a review see also [13] and references therein). However the generalization of the Hamiltonian reduction [14, 15] to supersymmetric WZNW models which leads to (super–) Toda models has been considered only for a restricted class of models, namely for those based on superalgebras which admit description in terms of fermionic simple roots only [16, 17]. Such Hamiltonian reduction has not been applied to supersymmetric WZNW models based on bosonic groups and supergroups which have bosonic simple roots in any root decomposition probably because of the lack of group–theoretical and physical grounds for imposing appropriate constraints on components of the WZNW currents, and also because of a simple argument that the Hamiltonian reduction should spoil supersymmetry of these models [18].

In the present paper we shall show how one can overcome this problem. The model considered below is an example of the relation between strings propagating in flat space–time and appropriately constrained WZNW models, while usually WZNW models are associated with strings propagating on group manifolds (see, for example, [19]), or coset spaces (see, for example, [20]). The WZNW group manifold under consideration is isomorphic to the structure group of the flat target superspace of the Green–Schwarz superstring. For $N = 2$, $D = 3$ target superspace of the Green–Schwarz superstring (as well as for $D = 3$ fermionic strings) this group is $Sl(2, \mathbf{R})$. This provides us with a physical motivation to study the Hamiltonian reduction of wider class of the supersymmetric WZNW models,

¹We denote the number of left and right supersymmetries on the worldsheet by small n , and by capital N the number of supersymmetries in target space

of which we shall consider in detail the simplest example, namely, the $n = (1, 1)$, $d = 2$ superconformal $Sl(2, \mathbf{R})$ WZNW model. An unusual feature we shall encounter with is that the nonlinear superfield constraints imposed for performing the Hamiltonian reduction are a mixture of first- and second-class constraints. Supersymmetry of the model is realized nonlinearly and is spontaneously broken. The set of independent currents which survive the Hamiltonian reduction contains (in the holomorphic sector) one bosonic current of spin 2 (which is the stress-tensor of the spin 0 Liouville mode) and two fermionic fields of spin $\frac{3}{2}$ and $-\frac{1}{2}$. The $n = 1$ superconformal system thus obtained is of the same kind as one used for describing a hierarchy of bosonic and fermionic strings embedded one into another [2]. This demonstrates from a somewhat different point a (classical) relation between the Green-Schwarz and Neveu-Schwarz-Ramond-type formulation of superstrings [21, 22, 23, 24].

The prescription to be used can be generalized to obtain super-Toda theories associated to any bosonic superaffine Lie algebra and for constructing corresponding supersymmetric extensions of ordinary \mathcal{W} -algebras together with their non-standard realizations. The study of these models may turn out to be useful, for instance, for getting new information about the structure of systems with spontaneously broken supersymmetry, their connection with the corresponding systems with linearly realized supersymmetry, and to gain deeper insight into the relationship between different superstring models. Some new structures can arise as a result of the Hamiltonian reduction of WZNW models based on superalgebras which contain simple bosonic roots.

The paper is organized as follows. In Section 2 we perform the Hamiltonian reduction of the supersymmetric $Sl(2, \mathbf{R})$ WZNW model and in Section 3 we demonstrate that the resulting system of equations is equivalent to that obtained in the $N = 2$, $D = 3$ Green-Schwarz superstring model [1]. Symmetry properties and field contents of the system are analysed in Section 4. In Section 5 we make the Hamiltonian analysis of the constraints and get (upon gauge fixing and constructing Dirac brackets) a Virasoro stress tensor and a supersymmetry current of an $n = 1$ superconformal theory which describes an $n = (1, 1)$, $D = 3$ fermionic string in a physical gauge where two longitudinal bosonic degrees of freedom of the string are gauge fixed. In Section 6 we discuss the general case of the Hamiltonian reduction of super-WZNW models based on the classical Lie groups to nonstandard supersymmetric versions of the Toda models. In Conclusion we discuss an outlook of applying the procedure to study more general class of models.

2 Supersymmetric $Sl(2, \mathbf{R})$ WZNW model and its reduction

The superfield action of the model in $n = (1, 1)$, $d = 2$ superspace (with flat Minkowski metric) parametrized by supercoordinates $Z = (z, \theta)$, $\bar{Z} = (\bar{z}, \bar{\theta})$ (where z, \bar{z} are bosonic and $\theta, \bar{\theta}$ are fermionic light-cone coordinates) has the form [9, 10]

$$S = \frac{k}{2} \int dz d\bar{z} d\theta d\bar{\theta} \left(\text{tr}(G^{-1} D G G^{-1} \bar{D} G) + \int dt \text{tr}(G^{-1} \frac{\partial}{\partial t} G \{G^{-1} D G, G^{-1} \bar{D} G\}) \right), \quad (1)$$

where $G(Z, \bar{Z})$ is a superfield taking its values on the $Sl(2, \mathbf{R})$ group manifold, $D = \frac{\partial}{\partial \theta} + i\theta \frac{\partial}{\partial z}$ and $\bar{D} = \frac{\partial}{\partial \bar{\theta}} + i\bar{\theta} \frac{\partial}{\partial \bar{z}}$ are supercovariant fermionic derivatives which obey the following anticommutation relations

$$\{D, D\} = 2i \frac{\partial}{\partial z}, \quad \{\bar{D}, \bar{D}\} = 2i \frac{\partial}{\partial \bar{z}}, \quad \{D, \bar{D}\} = 0, \quad (2)$$

and k is called the level of the WZNW model (for simplicity in what follows we put $k = 1$). The second (Wess–Zumino) term in (1) is an integral over $d = 3$, $n = 1$ superspace whose boundary is the (Z, \bar{Z}) superspace.

The action (1) is invariant under superconformal transformations of the $n = (1, 1)$, $d = 2$ superspace

$$Z' = Z'(Z), \quad \bar{Z}' = \bar{Z}'(\bar{Z});$$

$$D' = e^{-\Lambda} D, \quad \bar{D}' = e^{-\bar{\Lambda}} \bar{D}, \quad \text{where} \quad \Lambda = \log D\theta'(Z), \quad \bar{\Lambda} = \log \bar{D}\bar{\theta}'(\bar{Z}), \quad (3)$$

and under the superaffine $Sl(2, \mathbf{R})$ transformations of $G(Z, \bar{Z})$

$$G' = g_L^{-1}(Z) G(Z, \bar{Z}) g_R(\bar{Z}). \quad (4)$$

From (1) one gets the equations of motion

$$\bar{D}(D G G^{-1}) = 0, \quad D(G^{-1} \bar{D} G) = 0, \quad (5)$$

which read that the fermionic currents

$$\begin{aligned} \Psi(Z) &\equiv \frac{1}{i} D G G^{-1} = \psi(z) + \theta k(z) \\ &= \Psi^- E_- + \Psi^+ E_+ + \Psi^0 H, \\ \bar{\Psi}(\bar{Z}) &\equiv i G^{-1} \bar{D} G = \bar{\psi}(\bar{z}) + \bar{\theta} \bar{k}(\bar{z}) \\ &= \bar{\Psi}^- E_- + \bar{\Psi}^+ E_+ + \bar{\Psi}^0 H \end{aligned} \quad (6)$$

are (anti)chiral and hence conserved, and generate the superaffine $Sl(2, \mathbf{R})$ transformations (4). E_{\pm} , H are the $sl(2, \mathbf{R})$ algebra generators

$$[E_+, E_-] = H, \quad [H, E_{\pm}] = \pm 2E_{\pm}. \quad (7)$$

The supersymmetric WZNW model can be constrained by putting equal to zero the components of the currents (6) corresponding to the generators (7) of a vector subgroup of the $Sl(2, \mathbf{R})_L \times Sl(2, \mathbf{R})_R$ affine group. On this way one gets coset construction of superconformal field theories [11, 12]. Such constraints can be obtained from a gauged WZNW action [25], which has been under intensive study as an effective action for string configurations in target coset spaces (see, for example, [20] and references therein).

Another possibility of reducing WZNW models (studied below) is to impose constraints on those algebra-valued components of the WZNW currents which correspond to the nilpotent subalgebra of the positive (or negative) roots of the WZNW group generators [14, 17]. As we have already mentioned, this reduction establishes the relationship between (super) WZNW models [14, 17] and (super) Toda theories [26].

In the case under consideration one cannot perform such kind of Hamiltonian reduction in a straightforward way. Indeed, the basic supercurrents of the model are fermionic ², and the Hamiltonian reduction procedure prescribes putting equal to nonzero constants those (super)currents which correspond to the positive or negative simple roots of the WZNW (super)algebra (a reduction of this kind leads to so-called Abelian Toda models, see [14, 17]).

If a current component to be constrained is fermionic, it does not seem natural to put it equal to a Grassmann constant. A way to overcome this problem is to explicitly use in the constraints the Grassmann coordinates θ or/and $\bar{\theta}$ of superspace, however the resulting model is no longer supersymmetric [18].

Therefore we shall use an alternative possibility, which does not violate manifest supersymmetry. Firstly, we shall construct bosonic supercurrents out of fermionic ones (6) in an appropriate covariant way, and then impose constraints on the bosonic supercurrents. The natural candidates to be constrained are $sl(2)$ -valued components of the bosonic supercurrents

$$\begin{aligned} J &\equiv \partial G G^{-1} = J^+ E_+ + J^- E_- + J^0 H \\ \bar{J} &\equiv -G^{-1} \bar{\partial} G = \bar{J}^+ E_+ + \bar{J}^- E_- + \bar{J}^0 H. \end{aligned} \quad (8)$$

Together with the fermionic currents (6) the bosonic currents (8) constitute a differential one-superform being a pullback onto the $n = (1, 1)$, $d = 2$ superspace of the $Sl(2, \mathbf{R})$

² It deserves mentioning that the (super)algebra-valued supercurrents of a super-WZNW model have opposite statistic with respect to the corresponding (super)algebra generators.

Cartan form

$$\Omega = dGG^{-1}, \quad (9)$$

where

$$d = (dz - id\theta\theta)\frac{\partial}{\partial z} + (d\bar{z} - id\bar{\theta}\bar{\theta})\frac{\partial}{\partial \bar{z}} + d\theta D + d\bar{\theta}\bar{D}$$

is the external differential.³ By construction Eq. (9) satisfies the Maurer–Cartan equation (which is the zero curvature condition for the $Sl(2, \mathbf{R})$ connection form Ω)

$$d\Omega - \Omega \wedge \Omega = 0. \quad (10)$$

From (10) it follows that the bosonic currents (8) are composed out of the fermionic currents (6) as follows

$$\begin{aligned} J &= D\Psi - i\Psi\Psi = k(z) - i\psi(z)\psi(z) + i\theta(\partial\psi + \psi k - k\psi) \equiv j(z) + i\theta\chi(z), \\ \bar{J} &= \bar{D}\bar{\Psi} - i\bar{\Psi}\bar{\Psi} = \bar{k}(\bar{z}) - i\bar{\psi}(\bar{z})\bar{\psi}(\bar{z}) + i\bar{\theta}(\bar{\partial}\bar{\psi} + \bar{\psi}\bar{k} - \bar{k}\bar{\psi}) \equiv \bar{j}(\bar{z}) + i\bar{\theta}\bar{\chi}(\bar{z}). \end{aligned} \quad (11)$$

We are now in a position to constrain the supersymmetric $Sl(2, \mathbf{R})$ WZNW model the same way as was used to reduce the bosonic $Sl(2, \mathbf{R})$ WZNW model to the Liouville theory [14, 15]. We put equal to one the current components J^- and \bar{J}^+ of Eqs. (8), (11):

$$J^- = D\Psi^- + 2i\Psi^0\Psi^- = 1, \quad \bar{J}^+ = \bar{D}\bar{\Psi}^+ - 2i\bar{\Psi}^0\bar{\Psi}^+ = 1, \quad (12)$$

where $D + 2i\Psi^0$ and $\bar{D} - 2i\bar{\Psi}^0$ can be regarded as covariant derivatives with $\Psi^0, \bar{\Psi}^0$ being components of a connection form on $d = 2$ superspace. Eqs. (12) imply *nonlinear* relations between components of the fermionic currents (6)⁴.

By using the Gauss decomposition of the $Sl(2, \mathbf{R})$ group element

$$G(Z, \bar{Z}) = e^{\beta E_+} e^{\Phi H} e^{\gamma E_-} \quad (13)$$

one can (locally) write down the current components (6) in the following form

$$\begin{aligned} \Psi^- &= \frac{1}{i}D\gamma e^{-2\Phi}, & \Psi^+ &= \frac{1}{i}D\beta - \frac{1}{i}2\beta D\Phi - \beta^2\Psi^-, & \Psi^0 &= \frac{1}{i}D\Phi + \beta\Psi^-; \\ \bar{\Psi}^+ &= i\bar{D}\beta e^{-2\Phi}, & \bar{\Psi}^- &= i\bar{D}\gamma - 2i\gamma\bar{D}\Phi - \gamma^2\bar{\Psi}^+, & \bar{\Psi}^0 &= i\bar{D}\Phi + \gamma\bar{\Psi}^+; \end{aligned} \quad (14)$$

³Notice that, for instance, the $d\bar{\theta}$ component of Ω is $\Omega_{\bar{D}} = \frac{1}{i}G\bar{\Psi}G^{-1}$

⁴ $n=(2,2)$ superconformal WZNW models and current algebras with nonlinear constraints on fermionic currents, which formally have the same form as the r.h.s. of Eq. (11), were studied in [27, 28]. Though the group–theoretical motivation for imposing the constraints in this form is the same as in our case, the constraints of [27, 28] bear an essentially different meaning and result in unconstrained models in terms of $n=(1,1)$ superfields [27, 28].

Then the super-WZNW equations of motion (5) and the constraints (12) reduce to the following system of equations:

$$\bar{D}D\Phi = e^{2\Phi}\bar{\Psi}^+\Psi^-, \quad \bar{D}\Psi^- = 0 = D\bar{\Psi}^+; \quad (15)$$

$$D\Psi^- + 2D\Phi\Psi^- = 1, \quad \bar{D}\bar{\Psi}^+ + 2\bar{D}\Phi\bar{\Psi}^+ = 1. \quad (16)$$

Eqs. (15) correspond to the unconstrained supersymmetric WZNW model.

The chirality conditions for the Ψ^+ and $\bar{\Psi}^-$ component of (14) are identically satisfied provided that (15) are valid. Note also that neither $\beta(Z, \bar{Z})$ nor $\gamma(Z, \bar{Z})$ enters the Eqs. (15), (16) explicitly if Ψ^- and $\bar{\Psi}^+$ are considered as independent variables.

3 Connection with N=2, D=3 Green-Schwarz superstrings

In [1] the system of equations (15), (16) was obtained from the equations of motions and constraints describing a classical $N = 2$, $D = 3$ Green-Schwarz superstring in a doubly supersymmetric geometrical approach [3, 4, 5]. To demonstrate this point we have to make a digression on ideas of applying the geometrical approach to describing the dynamics of (super)-p-branes.

The geometrical approach implies the use of notions and methods of surface theory [29] to describe embedding of p-brane worldvolumes into target spaces for a purpose of solving for the constraints and reducing the p-brane equations of motion to nonlinear (sometimes exactly solvable) systems of equations for independent physical degrees of freedom of the p-branes. Its application to bosonic extended objects was initiated by Refs. [30, 31], and the reader may find a review of recent results in [32, 33, 34] and references therein.

The development of supersurface theory to treat supergravities as theories of embedding supersurfaces into target supermanifolds was carried out in [35].

A generalization of the geometrical approach to study super-p-branes stems from a doubly supersymmetric (so called twistor-like) formulation of these systems which is based on following principles (see [4] for details and references):

- i) the fermionic κ -symmetry of the Green-Schwarz formulation of super-p-branes [36, 21] is a manifestation of superdiffeomorphisms of worldvolume supersurfaces of the super-p-branes [37]. (This solves the problem of infinite reducibility of the κ -symmetry and makes it possible to carry out covariant Hamiltonian analysis of superstring dynamics at least on the classical level);
- ii) the geometrical ground for this is that the theory of super-p-branes is supposed to be a particular kind of doubly supersymmetric models (studied earlier in [38]) which

describe an embedding of supersurfaces into target superspaces. (This naturally incorporates twistor-like commuting spinors into the theory as superpartners of target superspace Grassmann coordinates).

One of the aims of the approach has been to push forward the problem of covariant quantization of superstring theory. As a result the methods to attain this objective have undergone substantial modifications and essential progress has been made during last few years (see [24] for a recent review).

At the same time the doubly supersymmetric formulation serves as a natural dynamical basis for generalizing methods of classical surface theory to study the embedding of supersurfaces corresponding to super-p-branes [4], and then one may try to apply the geometrical methods back to the analysis of variety of fundamental and solitonic super-p-branes we are having at hand at present time [39].

General properties of superstring and supermembrane worldvolumes embedded into flat target superspaces of various dimensions were studied in [4] (see also [40] for the case of $N=1$ superstrings). Using these results (which supergeneralize that of [31]) we demonstrate below how by specifying the embedding of worldsheet supersurface swept by the $N = 2$, $D = 3$ superstring one can solve for the Virasoro constraints and reduce superstring equations of motion to the super-Liouville-like system of equations (15), (16) [1]. Note that the classical equivalence of the twistor-like and Green-Schwarz formulation of the $N=2$, $D=3$ superstring was shown in [3].

In the doubly supersymmetric formulation worldsheet of the Green-Schwarz superstrings is a supersurface parametrized by two bosonic coordinates $z = (\tau + \sigma)$, $\bar{z} = (\tau - \sigma)$ and fermionic coordinates θ , $\bar{\theta}$ whose number should be equal to the number of independent κ -symmetry transformations in the standard Green-Schwarz formulation. In our case there are one left- and one right-handed Majorana-Weyl spinor coordinate $\theta, \bar{\theta}$, which means that we deal with $n = (1, 1)$ local supersymmetry on worldsheet supersurface

$$\mathcal{M}_{ws} : \quad \left(Z = (z, \theta), \quad \bar{Z} = (\bar{z}, \bar{\theta}) \right). \quad (17)$$

To describe intrinsic \mathcal{M}_{ws} geometry (i.e. $n=(1,1)$, $d=2$ supergravity) one should set on \mathcal{M}_{ws} a local frame of supervielbein one-forms which contains two bosonic vector and to fermionic spinor components

$$e^A(Z, \bar{Z}) = \left(e^a(Z, \bar{Z}), \quad e^\alpha(Z, \bar{Z}) \right), \quad (18)$$

where $a = (++, --)$ and $\alpha = (+, -)$ stand for light-cone vector and spinor indices of the $d = 2$ Lorentz group $SO(1, 1)$, respectively.

We consider an embedding of \mathcal{M}_{ws} into $N = 2$, $D = 3$ flat superspace-time parametrized by three bosonic vector and two Majorana spinor coordinates $X^{\underline{m}}(Z, \bar{Z})$, $\Theta^{1\mu}(Z, \bar{Z})$,

$\Theta^{2\mu}(Z, \bar{Z})$, where the underlined indices $\underline{m} = 0, 1, 2$ and $\underline{\mu} = 1, 2$ are vector and spinor indices of the $D = 3$ Lorentz group $SO(1, 2) \sim Sl(2, \mathbf{R})$, respectively.

A natural supersymmetric rigid frame in flat target superspace is

$$\Pi^{\underline{m}} = dX^{\underline{m}} + id\bar{\Theta}^i \Gamma^{\underline{m}} \Theta^i, \quad d\Theta^{i\mu} \quad (i = 1, 2). \quad (19)$$

$\Gamma_{\alpha\beta}^{\underline{m}}$ are $D = 3$ Dirac matrices.

The study of \mathcal{M}_{ws} embedding is started with fitting the rigid target-superspace frame (19) to that on the supersurface (18). To this end we transform (19) into a new local frame

$$E^{\underline{a}} = \Pi^{\underline{m}} u_{\underline{m}}^{\underline{a}}(X, \Theta), \quad E^{i\alpha} = d\Theta^{i\mu} v_{\underline{\mu}}^{\alpha}(X, \Theta), \quad (20)$$

by use of $u_{\underline{m}}^{\underline{a}}$ and $v_{\underline{\mu}}^{\alpha}$ matrices of the vector and spinor representation of the target-superspace Lorentz group $Sl(2, \mathbf{R})$, respectively. But since the vector and spinor components of (19) are subject to the Lorentz transformation simultaneously, u and v are not independent and connected through the well-known twistor-like relation

$$u_{\underline{m}}^{\underline{a}}(\Gamma^{\underline{m}})_{\underline{\mu}\underline{\nu}} = v_{\underline{\mu}}^{\alpha}(\Gamma^{\underline{a}})_{\alpha\beta} v_{\underline{\nu}}^{\beta}$$

between vectors and commuting spinors. This explains why the approach is called “twistor-like”.

From the general analysis of superstring dynamics in the twistor-like approach we learn [4] that the target-superspace local frame (20) can be attached to the supersurface as follows:

$$E^{\perp}(Z, \bar{Z}) = 0, \quad (21)$$

$$E^{\underline{a}}(Z, \bar{Z}) = \Pi^{\underline{m}} u_{\underline{m}}^{\underline{a}} = (dX^{\underline{m}} + id\bar{\Theta}^i \Gamma^{\underline{m}} \Theta^i) u_{\underline{m}}^{\underline{a}} = e^{\underline{a}}, \quad (22)$$

$$E^{1+}(Z, \bar{Z}) = d\Theta^{1\mu} v_{\underline{\mu}}^+ = e^+, \quad E^{2-}(Z, \bar{Z}) = d\Theta^{2\mu} v_{\underline{\mu}}^- = e^-, \quad (23)$$

where the target superspace indices split onto that of \mathcal{M}_{ws} and of the orthogonal vector direction ($\underline{a} \rightarrow (a, \perp)$; $\underline{\alpha} \rightarrow (+, -)$).

Eq. (21) tells us that one of the vector components of the target superspace frame can be made orthogonal to the supersurface (its pullback on \mathcal{M}_{ws} is zero) and three other relations (22), (23) identify (on \mathcal{M}_{ws}) components of the target superspace frame with the intrinsic \mathcal{M}_{ws} supervielbein components (18). This means that \mathcal{M}_{ws} geometry is induced by embedding.

From (21) and (22), using the orthogonality properties of $u_{\underline{m}}^{\underline{a}}$ ($u_{\underline{m}}^{\underline{a}} u_{\underline{n}\underline{a}} = \eta_{\underline{mn}}$), we find that the pullback on \mathcal{M}_{ws} of the vector one-superform $\Pi^{\underline{m}}(Z, \bar{Z}) = dX^{\underline{m}} + id\bar{\Theta}^i \Gamma^{\underline{m}} \Theta^i = e^{\underline{a}} u_{\underline{a}}^{\underline{m}}(Z, \bar{Z})$ is zero along the fermionic directions e^{α} of \mathcal{M}_{ws} :

$$DX^{\underline{m}} + iD\bar{\Theta}^i \Gamma^{\underline{m}} \Theta^i = 0, \quad \bar{D}X^{\underline{m}} + i\bar{D}\bar{\Theta}^i \Gamma^{\underline{m}} \Theta^i = 0, \quad (24)$$

where D, \bar{D} are covariant spinor derivatives on \mathcal{M}_{ws} which reduce to the flat derivatives (2) in a superconformal gauge for (18). Eq. (24) is called the geometrodynamical condition. Eqs. (21) and (22) ensure that the Virasoro constraints on superstring dynamics

$$\Pi_{++}^m \Pi_{++}^m = 0 = \Pi_{--}^m \Pi_{--}^m \quad (25)$$

are identically satisfied for such kind of embedding [4].

One can notice that Eqs. (21)–(23) are first-order differential equations on $X(Z, \bar{Z})$ and $\Theta(Z, \bar{Z})$. If they are solved one would know the shape of the worldsheet supersurface in $D = 3, N = 2$ superspace and thus would solve the problem of describing classical superstring motion. To solve (21)–(23) one must know $v_{\underline{\mu}}^{\alpha}(Z, \bar{Z})$ and the components of $e^a(Z, \bar{Z}), e^{\alpha}(Z, \bar{Z})$. To get this information one should study the integrability conditions of (21)–(23) which are obtained by taking the external differential of (21)–(23). (This is the general strategy one pursues in the geometrical approach [30]–[32], [4]). Basic integrability conditions thus obtained are [4, 5]:

$$de^a - \Omega_b^a e^b = ie^{\alpha} \gamma_{\alpha\beta}^a e^{\beta} = T^a, \quad (26)$$

$$\Omega^{\perp a} = K_b^a e^b + K_{\alpha}^a e^{\alpha}, \quad (27)$$

where $\gamma_{\alpha\beta}^a$ are d=2 Dirac matrices, and external differentiation and product of the forms are implied. Eqs. (26), (27) contain one forms $\Omega_b^a, \Omega^{\perp a}$ which are Cartan forms of the $Sl(2, \mathbf{R})$ Lorentz group (9) constructed out of the matrix $v_{\underline{\mu}}^{\alpha}$ components:

$$\Omega^{ab} = \epsilon^{ab} v_{\underline{\mu}}^+ dv^{-\underline{\mu}}, \quad \Omega^{\perp a} = \gamma_{\alpha\beta}^a v_{\underline{\mu}}^{\alpha} dv^{\beta \underline{\mu}}. \quad (28)$$

Eq. (26) determines parallel transport of vector supervielbeins along \mathcal{M}_{ws} carried out by induced connection Ω^{ab} . It reads that the connection possesses torsion whose spinor-spinor components are constrained to be equal to the γ -matrix components. This is a basic torsion constraint of any supergravity theory. In the geometrical approach it is not imposed by hand, but appears, together with other $n = (1, 1), d = 2$ supergravity constraints [41], as a consistency condition of \mathcal{M}_{ws} embedding (21)–(23) [5, 4]. From [41] we know that d=2 supergravity thus constrained is superconformally flat. This means that by gauge fixing \mathcal{M}_{ws} superdiffeomorphisms one can choose a superconformal gauge where the supervielbeins (18) take the form

$$\begin{aligned} e^{++} &= e^{-2(\Phi-L)}(dz - id\theta\theta), & e^{--} &= e^{-2(\Phi+L)}(d\bar{z} - id\bar{\theta}\bar{\theta}), \\ e^{+} &= e^{-(\Phi-L)}(d\theta + iD\Phi(dz - id\theta\theta)), & e^{-} &= e^{-(\Phi+L)}(d\bar{\theta} + i\bar{D}\Phi(d\bar{z} - id\bar{\theta}\bar{\theta})), \end{aligned} \quad (29)$$

where the superfield $\Phi(Z, \bar{Z})$ corresponds to Weyl rescaling and $L(Z, \bar{Z})$ corresponds to local Lorentz $SO(1, 1)$ transformations of the supervielbeins. Note that Weyl transformations are not a symmetry of the model, while the local $SO(1, 1)$ transformations are [4].

The second condition (27) specifies the expansion of $\Omega^{\perp a}$ in the \mathcal{M}_{ws} supervielbein components: the bosonic matrix $K_{ab}(Z)$ is symmetric and has the properties of the second fundamental form analogous to that of the bosonic surfaces, while spinor components $K_\alpha(Z) \equiv K_\beta^a \gamma_{a\alpha}^\beta$ of the Grassmann-odd spin-tensor K_β^a can be associated with a fermionic counterpart of the second fundamental form along Grassmann directions of the supersurface [4].

By construction the Cartan forms (28) must satisfy the $Sl(2, \mathbf{R})$ Maurer–Cartan equations (10) which split into two systems of equations with respect to the worldsheet indices:

$$d\Omega^{\perp a} - \Omega_b^a \Omega^{\perp b} = 0, \quad (30)$$

$$R^{ab} = d\Omega^{ab} = \Omega^{\perp a} \Omega^{\perp b}. \quad (31)$$

Eq. (30) is known as the Codazzi equation and (31) is called the Gauss equation in surface theory. On the other hand one can recognize in (30) and (31) relations which specify geometry on a two-dimensional coset space $\frac{SO(1,2)}{SO(1,1)}$ with $\Omega^{\perp a}$ being a vielbein, Ω^{ab} being a spin connection and R^{ab} being a constant curvature tensor of $\frac{SO(1,2)}{SO(1,1)}$.

Thus we have reduced the problem of studying superstring dynamics (as the embedding of a supersurface into target superspace) to study a mapping of \mathcal{M}_{ws} onto the bosonic coset space of constant curvature. It is here that a connection of Green–Schwarz superstring dynamics with the $n = (1, 1)$ super–WZNW model comes out.

To completely describe superstring dynamics in geometrical terms we should specify what additional conditions on embedding arise when the superstring equations of motion are taken into account [30, 31, 32, 4]. In bosonic surface theory such an embedding is called minimal and is characterized by traceless second fundamental form K_{ab} . In the supersymmetric case we get analogous condition on the bosonic part of the second fundamental form (27) which is in one to one correspondence with $X^{\underline{m}}$ equations of motion

$$K_a^{\underline{a}} \equiv D_a \left(D^{\underline{a}} X^{\underline{m}} - i \bar{\Theta}^i \Gamma^{\underline{m}} D^a \Theta^i \right) u_{\underline{m}}^{\perp} = 0, \quad (32)$$

where $D_a = (D_{--}, D_{++})$ is a vector covariant derivative, which reduces to $(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})$ in the superconformal gauge. In addition, $\Theta(Z, \bar{Z})$ equations of motion, which in the twistor-like approach have the form

$$K_- \equiv D_{--} \Theta^{1\mu} v_{\underline{\mu}}^- = 0, \quad K_+ \equiv D_{++} \Theta^{2\mu} v_{\underline{\mu}}^+ = 0,$$

result in vanishing the fermionic counterpart of the second fundamental form in Eq. (27), namely

$$K_\alpha(Z, \bar{Z}) \equiv K_\beta^a \gamma_{a\alpha}^\beta = 0, \quad \alpha = (+, -). \quad (33)$$

Thus we see that in the geometrical approach the dynamical string equations of motion are replaced by algebraic conditions (constraints) on components of the Cartan form (27), and the role of the dynamical equations is taken by the Maurer–Cartan equations (30), (31) (which reminds the reduction of the WZNW model).

The minimal embedding conditions (32), (33) further reduce the number of independent superfields which determine the induced geometry on the worldsheet supersurface \mathcal{M}_{ws} . One can show [1, 40] that in the superconformal gauge (29) $\Omega^{\perp a}$ and Ω^{ab} (which now, by virtue of theorems of surface theory, bear all information about \mathcal{M}_{ws}) depend only on two bosonic superfields $L(Z, \bar{Z})$ and $\Phi(Z, \bar{Z})$ (29), and two fermionic superfields $\bar{\Psi}^+(\bar{Z})$ and $\Psi^-(Z)$. The leading components of $\Phi(Z, \bar{Z})$, $\bar{\Psi}^+(\bar{Z})$ and $\Psi^-(Z)$ describe one bosonic and two fermionic physical degrees of freedom of the classical $N=2$, $D=3$ Green–Schwarz superstring. As to the superfield $L(Z, \bar{Z})$, we have already mentioned that it corresponds to purely gauge degrees of freedom reflecting local worldsheet $SO(1,1)$ invariance of the superstring model. The exact form of the basic fermionic components of the Cartan forms (28) is [1] (in our convention)

$$\begin{aligned} F^- &\equiv \frac{1}{i} v_\mu^- Dv^{-\mu} = \Psi^- e^{\Phi-L}, & F^+ &\equiv \frac{1}{i} v_\mu^+ Dv^{+\mu} = 0, & F^0 &\equiv \frac{1}{i} v_\mu^+ Dv^{+\mu} = \frac{1}{2i}(D\Phi + DL); \\ \bar{F}^+ &\equiv i v_\mu^+ \bar{D}v^{+\mu} = \bar{\Psi}^+ e^{\Phi+L}, & \bar{F}^- &\equiv i v_\mu^- \bar{D}v^{-\mu} = 0, & \bar{F}^0 &\equiv i v_\mu^+ \bar{D}v^{+\mu} = \frac{i}{2}(\bar{D}\Phi - \bar{D}L). \end{aligned} \quad (34)$$

The bosonic components of the Cartan forms (28) are not independent and expressed in terms of (34) (similar to (11)) through the Maurer–Cartan equations (10) (or (30), (31)), and the superfields (34) themselves obey the $n = (1,1)$ superconformal invariant equations (15), (16) which now follow from (31) and (30) [1]. One should not confuse the Liouville system describing classical physical modes of the string with (super)Liouville modes emerging as anomalies of quantized noncritical strings [7]. This completes our sketch of how Eqs. (15), (16) emerge in the $N = 2$, $D = 3$ superstring model.

Note that in the process of reducing the problem of describing classical $N = 3$, $D = 3$ superstring dynamics to the Codazzi–Gauss equations (30), (31) we have hidden $N = 2$, $D = 3$ space–time supersymmetry since the \mathcal{M}_{ws} differential one–forms (28) take their values in the bosonic $sl(2, \mathbf{R})$ algebra and describe \mathcal{M}_{ws} mapping onto the bosonic coset space. Thus in the geometrical approach Green–Schwarz superstrings look much more like fermionic strings which *a priori* propagate in bosonic space–time. We shall turn back to this point in Section 5.

Let us now compare the $Sl(2, \mathbf{R})$ Cartan form components (34) with the $Sl(2, \mathbf{R})$ WZNW fermionic currents of the previous section. We see that the components of (34) do not have the form of the Gauss decomposition, they are nonchiral and, hence, differ from the currents (14). However, F , \bar{F} and Ψ , $\bar{\Psi}$ relate to each other the same way as a zero curvature connection of a bosonic Toda theory [26] relates to (anti)chiral currents of the corresponding reduced WZNW model [15]. To establish the relationship between Eqs. (34) and Eqs. (14) one should perform in (34) an independent left and right gauge transformation of $F = F^- E_- + F^0 H + F^+ E_+$ and $\bar{F} = \bar{F}^- E_- + \bar{F}^0 H + \bar{F}^+ E_+$, which correspond, respectively, to the $\beta(Z, \bar{Z})$ and $\gamma(Z, \bar{Z})$ factor of the Gauss decomposition (13) of an $Sl(2, \mathbf{R})$ group element:

$$\hat{F} = \frac{1}{i} D\beta E_+ + e^{\beta E_+} F e^{-\beta E_+}, \quad \hat{\bar{F}} = \frac{1}{i} \bar{D}\beta E_+ + e^{\beta E_+} \bar{F} e^{-\beta E_+}, \quad (35)$$

$$\hat{F} = i D\gamma E_- + e^{-\gamma E_-} F e^{\gamma E_-}, \quad \hat{\bar{F}} = i \bar{D}\gamma E_- + e^{-\gamma E_-} \bar{F} e^{\gamma E_-}. \quad (36)$$

To perform an appropriate β -transformation (35) we first gauge fix $L = \Phi$ in (34), and then choose $\beta(Z, \bar{Z})$ to satisfy the condition

$$\bar{D}\beta = \frac{1}{i} \bar{\Psi}^+ e^{2\Phi}.$$

Then the $\hat{\bar{F}}$ components in (35) turn to zero and \hat{F}^\mp , \hat{F}^0 coincide with Ψ^\mp , Ψ^0 , respectively.

On the other hand, to perform an appropriate γ -transformation we choose in (34) $L = -\Phi$ and take $\gamma(Z, \bar{Z})$ to satisfy

$$D\gamma = i \Psi^- e^{2\Phi}.$$

Then the \hat{F} components of (36) turn to zero and \hat{F}^\pm , \hat{F}^0 reduce to $\bar{\Psi}^\pm$, $\bar{\Psi}^0$. Notice that the gauge transformation parameters $\beta(Z, \bar{Z})$ and $\gamma(Z, \bar{Z})$ indeed coincide with those of Eqs. (14).

To conclude this section we should point to a problem which we have not solved. It is the problem of constructing a gauged version of the WZNW action (1) from which the constraints (12) could be obtained as the equations of motion of auxiliary gauge fields. The straightforward application of the procedure used for gauging the WZNW models subject to ‘standard’ Hamiltonian reduction [14, 15, 17] does not work, since in our case the constraints (12) are nonlinear, contain supercovariant derivatives of fermionic currents and, as we shall see in the next section, are a mixture of first- and second-class constraints. However an indirect indication that such a gauged supersymmetric WZNW action may exist is that there exist versions of the $N = 2$, $D = 3$ superstring action with

local $n = (1, 1)$ worldsheet supersymmetry [3, 5] from which one can get the system of equations (15), (16). As we have demonstrated, the currents that arise in the superstring model are connected with the WZNW currents by the local transformations (35), (36), which might be associated with local symmetry transformations of the gauged WZNW action if the latter existed.

4 Superconformal properties of the model and of the related Liouville system

Let us consider now superconformal properties of fields subject to the constraints (16). The Eqs. (15), (16) are invariant under the $n = (1, 1)$ transformations (3) provided that the latter are accompanied by the following left–right H –rotation of $G(Z, \bar{Z})$ (13):

$$\hat{G} = e^{-(\Lambda + \frac{1}{2}D\Lambda\Psi^-)H} G e^{-(\bar{\Lambda} + \frac{1}{2}\bar{D}\bar{\Lambda}\bar{\Psi}^+)H}. \quad (37)$$

Then, due to (14),

$$\hat{\Phi} = \Phi - (\Lambda + \bar{\Lambda}) - \frac{1}{2}(D\Lambda\Psi^- + \bar{D}\bar{\Lambda}\bar{\Psi}^+), \quad \hat{\beta} = e^{-2\Lambda - D\Lambda\Psi^-}\beta, \quad \hat{\gamma} = e^{-2\bar{\Lambda} - \bar{D}\bar{\Lambda}\bar{\Psi}^+}\gamma, \quad (38)$$

and

$$\hat{\Psi}^- = e^\Lambda\Psi^-, \quad \hat{\Psi}^+ = e^{-3\Lambda - D\Lambda\Psi^-}\Psi^+, \quad \hat{\Psi}^0 = e^{-\Lambda}(\Psi^0 - \frac{1}{i}D(\Lambda + \frac{1}{2}D\Lambda\Psi^-)), \quad (39)$$

$$\hat{\bar{\Psi}}^+ = e^{\bar{\Lambda}}\bar{\Psi}^+, \quad \hat{\bar{\Psi}}^- = e^{-3\bar{\Lambda} - \bar{D}\bar{\Lambda}\bar{\Psi}^+}\bar{\Psi}^-, \quad \hat{\bar{\Psi}}^0 = e^{-\bar{\Lambda}}(\bar{\Psi}^0 - i\bar{D}(\bar{\Lambda} + \frac{1}{2}\bar{D}\bar{\Lambda}\bar{\Psi}^+)). \quad (40)$$

From Eqs. (38) – (40) we see that the supercurrents Ψ^+ and $\bar{\Psi}^-$ transform nonlinearly under the modified $n = (1, 1)$ superconformal transformations, and even though the superfields Φ , Ψ^- , $\bar{\Psi}^+$, Ψ^0 and $\bar{\Psi}^0$ transform linearly, the $n = (1, 1)$ superconformal symmetry is nonlinearly realized on the components of the superfields, since they are constrained by Eqs. (16) (the basic fields belong in fact to different supermultiplets).

The general solution to (16) (obtained by taking into account (anti)chirality of Ψ^- , $\bar{\Psi}^+$) is ([1]):

$$\Psi^- = \psi^-(z) + \theta(1 + i\psi^-\partial\psi^-), \quad \bar{\Psi}^+ = \bar{\psi}^+(\bar{z}) + \bar{\theta}(1 + i\bar{\psi}^+\bar{\partial}\bar{\psi}^+), \quad (41)$$

$$\Phi = \phi(z, \bar{z}) + \frac{i}{2}\theta e^{-2\phi}\partial(e^{2\phi}\psi^-) + \frac{i}{2}\bar{\theta} e^{-2\phi}\bar{\partial}(e^{2\phi}\bar{\psi}^+) + \theta\bar{\theta}\psi^-\bar{\psi}^+\partial\bar{\partial}\phi. \quad (42)$$

From Eqs. (3), (38) – (42) it follows that under infinitesimal (anti)holomorphic supersymmetry transformations $\theta \rightarrow \theta - \epsilon(z)$, $\bar{\theta} \rightarrow \bar{\theta} - \bar{\epsilon}(\bar{z})$ the leading component of Φ transforms as follows

$$\phi(z, \bar{z}) \rightarrow \phi(z, \bar{z}) + i\epsilon(z)\psi^-\partial\phi + \frac{i}{2}\partial(\epsilon(z)\psi^-) + i\bar{\epsilon}(\bar{z})\bar{\psi}^+\bar{\partial}\phi + \frac{i}{2}\bar{\partial}(\bar{\epsilon}(\bar{z})\bar{\psi}^+), \quad (43)$$

and the leading components of Ψ^- , $\bar{\Psi}^+$ transform as Goldstone fermions [42]

$$\psi^-(z) \rightarrow \psi^-(z) + \epsilon(z) + i\epsilon(z)\psi^-\partial\psi^-, \quad \bar{\psi}^+(\bar{z}) \rightarrow \bar{\psi}^+(\bar{z}) + \bar{\epsilon}(\bar{z}) + i\bar{\epsilon}(\bar{z})\bar{\psi}^+\partial\bar{\psi}^+, \quad (44)$$

which signifies that the $n = (1, 1)$ supersymmetry is spontaneously broken. Note that the form of the transformations (44) does not imply that $\psi^-(z)$, $\bar{\psi}^+(\bar{z})$ are pure gauge degrees of freedom, since here superconformal symmetry is not a local symmetry in a full sense and cannot be used for reducing the number of the physical degrees of freedom of the model (there is no first-class constraints which generate the superconformal symmetry).

The supersymmetry transformation properties of the leading components of Ψ^+ and Ψ^0 ($\bar{\Psi}^-$ and $\bar{\Psi}^0$ in the antiholomorphic sector) are

$$\begin{aligned} \delta\psi^+ &= (j^+ + 2i\psi^0\psi^+)\epsilon + i\psi^-\psi^+\partial\epsilon, & \delta\psi^0 &= (j^0 + i\psi^+\psi^-)\epsilon + \frac{1}{2}(1 + 2i\psi^-\psi^0)\partial\epsilon, \\ \bar{\delta}\bar{\psi}^- &= (\bar{j}^- + 2i\bar{\psi}^0\bar{\psi}^-)\bar{\epsilon} + i\bar{\psi}^+\bar{\psi}^-\partial\bar{\epsilon}, & \bar{\delta}\bar{\psi}^0 &= (\bar{j}^0 + i\bar{\psi}^-\bar{\psi}^+)\bar{\epsilon} + \frac{1}{2}(1 + 2i\bar{\psi}^+\bar{\psi}^0)\partial\bar{\epsilon}. \end{aligned} \quad (45)$$

Under conformal transformations ($z \rightarrow z - \lambda(z)$, $\bar{z} \rightarrow \bar{z} - \bar{\lambda}(\bar{z})$), the fields ϕ , ψ^- , $\bar{\psi}^+$ transform as follows:

$$\delta\phi = \lambda\partial\phi + \frac{1}{2}\partial\lambda + \bar{\lambda}\bar{\partial}\bar{\phi} + \frac{1}{2}\bar{\partial}\bar{\lambda}, \quad (46)$$

$$\delta\psi^- = \lambda\partial\psi^- - \frac{1}{2}\psi^-\partial\lambda, \quad \bar{\delta}\bar{\psi}^+ = \bar{\lambda}\bar{\partial}\bar{\psi}^+ - \frac{1}{2}\bar{\psi}^+\bar{\partial}\bar{\lambda}, \quad (47)$$

and

$$\delta\psi^+ = \lambda\partial\psi^+ + \frac{3}{2}\psi^-\partial\lambda, \quad \delta\psi^0 = \lambda\partial\psi^0 + \frac{1}{2}\psi^0\partial\lambda + \frac{1}{4}\psi^-\partial^2\lambda, \quad (48)$$

$$\bar{\delta}\bar{\psi}^- = \bar{\lambda}\bar{\partial}\bar{\psi}^- + \frac{3}{2}\bar{\psi}^+\bar{\partial}\bar{\lambda}, \quad \bar{\delta}\bar{\psi}^0 = \bar{\lambda}\bar{\partial}\bar{\psi}^0 + \frac{1}{2}\bar{\psi}^0\bar{\partial}\bar{\lambda} + \frac{1}{4}\bar{\psi}^+\bar{\partial}^2\bar{\lambda}. \quad (49)$$

The transformations (43) – (49) form an $n = (1, 1)$ superconformal algebra which closes on the mass shell. For instance

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_\lambda, \quad (50)$$

where $\lambda = 2i\epsilon_1\epsilon_2$. From (47) we see that ψ^- , $\bar{\psi}^+$ have conformal spin $-\frac{1}{2}$, and ψ^+ , $\bar{\psi}^-$ have spin $\frac{3}{2}$. As to the fields ψ^0 , $\bar{\psi}^0$, their transformation properties (45), (48), (49) reflect the fact that, due to the constraints (12), these fields are not independent (see next Section for the details).

Comparing (43) with (46) one can notice that these transformations have the same form for $\lambda = i\epsilon\psi^-$ and $\bar{\lambda} = i\bar{\epsilon}\bar{\psi}^+$. Thus, in some sense, supersymmetric transformations of ϕ can be hidden in its conformal transformations. This effect was observed by the authors of [9] in the unconstrained supersymmetric $O(N)$ WZNW model. Note also that the

supersymmetric transformations of ψ^- , $\bar{\psi}^+$ contain only the fermionic fields themselves, nevertheless, as we have seen (Eq. (50)), the commutator of two supersymmetry transformations leads to a conformal transformation, therefore the superconformal symmetry of our model is fully fledged in contrast to a Grassmann symmetry of a free fermion system of Ref. [9].

Finally, we present the component equations resulting from (15), (16) [1]:

$$\partial\bar{\partial}\phi = -e^{2\phi}, \quad (51)$$

$$\bar{\partial}\psi^- = 0, \quad \partial\bar{\psi}^+ = 0. \quad (52)$$

As one might expected, the reduction of the supersymmetric $Sl(2, \mathbf{R})$ WZNW model resulted in the Liouville equation (51), which is accompanied by the free chiral fermion equations (52). An interesting point is that the whole system of the equations (51), (52) is invariant under the nonlinear $n = (1, 1)$ superconformal transformations (43), (44), and thus can be considered as a supersymmetric generalization of the Liouville equation alternative [1] to the conventional one based on the supergroup $OSp(1|2)$ [6].⁵ Together with this system one should consider free fermionic fields $\psi^+(z)$, $\bar{\psi}^-(z)$ of spin $\frac{3}{2}$ which also remain independent after the Hamiltonian reduction. The properties of the whole system are studied in the next Section.

5 Current algebra, Hamiltonian analysis of the constraints and connection with fermionic strings

Poisson brackets of the fermionic superfield currents (6) generate two (anti)commuting copies of the superaffine $SL(2, \mathbf{R})$ algebra [17]

$$\{tr(A\Psi(X)), tr(B\Psi(Y))\}_{PB} = -\delta(X - Y)tr[A, B]\Psi(Y) - iD_X\delta(X - Y)tr(AB), \quad (53)$$

where A , B stand for E_+ , E_- and H ; $X = (z_1, \theta_1)$, $Y = (z_2, \theta_2)$,

$\delta(X - Y) = \delta(z_1 - z_2)(\theta_1 - \theta_2)$, and $D_X = \frac{\partial}{\partial\theta_1} + i\theta_1\frac{\partial}{\partial z_1}$.

It is implied that the Poisson brackets are equal-time, i.e. $z_1 + \bar{z}_1 = z_2 + \bar{z}_2$. Since the Poisson brackets of $\bar{\Psi}(\bar{Z})$ have the same form as (53), we restrict ourselves to the consideration of only the holomorphic sector of the model.

⁵In [1] there was an overstatement that the two super-Liouville systems are not connected with each other by local redefinition of fields. In fact, this is only true one way, i.e. when one tries to get the free fermion equations from the standard super-Liouville equations [6]. But it turns out possible to locally express the standard super-Liouville fields through the fields of (51), (52) [43].

From (53) one can get the Poisson brackets of the bosonic superfield currents $J(Z)$ (11). We do not present them explicitly because of a somewhat cumbersome structure of their r.h.s. Instead we shall deal with Poisson brackets of the ordinary field components of $\Psi(Z)$ (6) and $J(Z)$ (11). It is convenient to work with $\psi(z)$ and $j(z)$ as the independent components of the superfield currents, since they are completely decoupled [9, 10]:

$$\begin{aligned}
\{j^+(z_1), j^-(z_2)\}_{PB} &= 2\delta(z_1 - z_2)j^0(z_2) - \partial\delta(z_1 - z_2), \\
\{j^0(z_1), j^\pm(z_2)\}_{PB} &= \pm\delta(z_1 - z_2)j^\pm(z_2), \\
\{j^0, j^0\}_{PB} &= -\frac{1}{2}\partial\delta(z_1 - z_2) \\
\{j^+, j^+\}_{PB} &= \{j^-, j^-\}_{PB} = 0, \\
\{j^a, \psi^b\}_{PB} &= 0, \\
\{\psi^+(z_1), \psi^-(z_2)\}_{PB} &= -i\delta(z_1 - z_2), \\
\{\psi^0(z_1), \psi^0(z_2)\}_{PB} &= -\frac{i}{2}\delta(z_1 - z_2), \\
\{\psi^+, \psi^+\}_{PB} &= \{\psi^-, \psi^-\}_{PB} = 0.
\end{aligned} \tag{54}$$

Using (54) it is rather easy to understand the structure of the superfield constraints (12), which, in components, reduce to

$$j^- - 1 = 0, \quad \psi^0 - j^0\psi^- - \frac{1}{2}\partial\psi^- = 0 \tag{55}$$

(in the holomorphic sector).

A linear combination of the constraints (55)

$$C_F = \psi^0 - j^0\psi^- - \frac{1}{2}\partial\psi^- + \frac{1}{4}\partial\psi^-(j^- - 1) = 0, \tag{56}$$

$$C_B = j^-(1 + 2i\psi^-C_F) - 1 = 0 \tag{57}$$

have the following form of the Poisson brackets (54):

$$\{C_B, C_B\}_{PB} = \{C_B, C_F\}_{PB} = 0, \tag{58}$$

$$\{C_F(z_1), C_F(z_2)\}_{PB} = -\frac{i}{2}\delta(z_1 - z_2). \tag{59}$$

Note that the equations (56) – (59) are satisfied in a weak sense, i.e. $C_B(Z)$ and $C_F(Z)$ can be put to zero only upon calculating their Poisson brackets with other field expressions. From (58), (59) we conclude that the bosonic constraint (57) is of the first class and the fermionic constraint (56) is of the second class. The latter can be put to zero in the strong sense by replacing the Poisson brackets with Dirac brackets [44]

$$\{f, g\}^* = \{f, g\}_{PB} - 2i \int \int dx dy \{f, C_F(x)\}_{PB} \delta(x - y) \{C_F(y), g\}_{PB}, \tag{60}$$

where $f(z)$ and $g(z)$ are arbitrary phase-space functions.

Upon introducing the Dirac brackets one can eliminate ψ^0 (and $\bar{\psi}^0$ in the antiholomorphic sector) from the number of the physical variables of the model.

Thus, within the course of the Hamiltonian reduction of the model we have encountered an unusual situation, namely, the appearance of the second-class constraint. Usually, when performing the Hamiltonian reduction of WZNW models, one restricts oneself to imposing first-class constraints [14, 15, 17].

A peculiar feature of the case under consideration is that though superfield constraints (12) are a mixture of the first- and second-class constraints, after the superconformal transformations (3) the first-class constraint component (57) of (12) remains in the first class.

The bosonic first-class constraint (58) reflects the existence of the invariance of the model under the chiral transformations $g^+(Z)$ corresponding to the E_+ subalgebra of the $sl(2, \mathbf{R})$ algebra (7). We should stress that the parameter of these transformations is a chiral *superfield*, since the whole superfield constraint $J^-(Z) - 1 = 0$ in (11) commutes (under (53)) with the superfield current $\Psi^-(Z)$ being the generator of $g^+(Z)$. A subtle point of the model at hand is that the presence of only one (bosonic) first-class constraint (57) (while its superpartner (56) is of the second class) indicates that only the bosonic $g^+(Z)|_{\theta=0}$ -transformations form the real gauge symmetry of the model which can be subject to further gauge fixing. The fermionic $Dg^+(Z)|_{\theta=0}$ -transformations (as the supersymmetry transformations (43)–(45)) are not fully fledged local transformations and cannot be used to reduce the number of physical degrees of freedom of the model. Note also that $\psi^+(z)$ transforms as a Goldstone field under $Dg^+(Z)|_{\theta=0}$:

$$\delta\psi^+(z) = -iDg^+(Z)|_{\theta=0}. \quad (61)$$

To analyse the superconformal structure of the physical sector of the model we impose an additional condition which fixes the $g^+(Z)|_{\theta=0}$ gauge transformations and converts (57) into a second-class constraint. Namely, as in the case of bosonic WZNW models [14] we impose the Drinfeld–Sokolov gauge [45]:

$$J^0|_{\theta=0} = j^0 = 0. \quad (62)$$

The constraint (62) is invariant under superconformal transformations of the supercurrent J^0 (8) provided the superconformal transformations are accompanied by a $g^+(Z)$ -transformation whose leading component depends on superconformal parameters $\epsilon(z)$, $\lambda(z)$ and supercurrent components. The explicit expression for $g^+(Z)|_{\theta=0}$ is derived from the $J^0(Z)$ transformation law, which is obtained from (37)–(39) and (11):

$$\hat{j}^0 = e^{-2\Lambda}(J^0 - iD\Lambda\Psi^0) + e^{-\Lambda-\frac{1}{2}D\Lambda\Psi^-}\partial e^{-\Lambda} - e^{-2\Lambda}\partial e^{-\frac{1}{2}D\Lambda\Psi^-} + e^{-2\Lambda-D\Lambda\Psi^-}g^+(Z). \quad (63)$$

Having in mind that $\hat{J}^0|_{\theta=0} = J^0|_{\theta=0} = 0$ we get for the infinitesimal transformations

$$g^+(Z)|_{\theta=0} = \partial^2 \lambda(z) + i\epsilon(\psi^+ + \frac{1}{2}\partial^2 \psi^- + 2i\psi^+ \psi^- \partial \psi^- - 2\psi^- j^+) - \frac{i}{2}\partial \epsilon \partial \psi^-. \quad (64)$$

To treat the conditions (57), (62) in the strong sense we should use them to construct new Dirac brackets with respect to which Eqs. (57), (62) commute. For this to be achieved in a simplest way one should find a pair of second-class constraints, being a combination of (57) and (62), which commute in a canonical way with respect to the Dirac brackets (60). Their form turns out to be as follows:

$$\begin{aligned} C_B^1 &= (j^- - 1)(1 - \frac{i}{2}\psi^- \partial \psi^-) = 0, \\ C_B^2 &= j^0 + \frac{1}{4}\partial C_B^1(1 + i\psi^- \partial \psi^-) = 0. \\ \{C_B^1(z_1), C_B^2(z_2)\}^* &= \delta(z_1 - z_2), \quad \{C_B^1(z_1), C_B^1(z_2)\}^* = \{C_B^2(z_1), C_B^2(z_2)\}^* = 0. \end{aligned} \quad (65)$$

Using (65) we construct new Dirac brackets

$$\begin{aligned} \{f, g\}^{**} &= \{f, g\}^* - \int \int dx dy \{f, C_B^1(x)\}^* \delta(x - y) \{C_B^2(y), g\}^* \\ &\quad + \int \int dx dy \{f, C_B^2(x)\}^* \delta(x - y) \{C_B^1(y), g\}^*, \end{aligned} \quad (66)$$

which allow one to treat the constraints (65) in the strong sense. Under the Dirac brackets (66) the fields $\psi^+(z)$, $\psi^-(z)$ and $j^+(z)$ have the following commutation properties:

$$\begin{aligned} \{\psi^+(z_1), \psi^+(z_2)\}^{**} &= -\frac{ik}{2}\partial^2 \delta(z_1 - z_2), \\ \{\psi^+(z_1), \psi^-(z_2)\}^{**} &= -ik\delta(z_1 - z_2), \\ \{\psi^-(z_1), \psi^-(z_2)\}^{**} &= \{j^+(z_1), \psi^+(z_2)\}^{**} = \{j^+(z_1), \psi^-(z_2)\}^{**} = 0, \\ \{j^+(z_1), j^+(z_2)\}^{**} &= \frac{k}{2}\partial^3 \delta(z_1 - z_2) + \delta(z_1 - z_2)\partial j^+(z_2) - 2\partial \delta(z_1 - z_2)j^+(z_2). \end{aligned} \quad (67)$$

In Eqs. (67), (68) and below we restored the explicit dependence of quantities on the level k of the WZNW model. The commutation relations (67) can be simplified even further if instead of ψ^+ one considers

$$b(z) = \psi^+(z) - \frac{1}{4}\partial^2 \psi^-(z), \quad c(z) = \frac{1}{k}\psi^-(z) \quad (69)$$

as the independent fermionic fields. Then $b(z)$ and $c(z)$ are canonical conjugate free fields with spin $\frac{3}{2}$ and $-\frac{1}{2}$, respectively:

$$\{b(z_1), c(z_2)\}^{**} = -i\delta(z_1 - z_2), \quad \{b(z_1), b(z_2)\}^{**} = \{c(z_1), c(z_2)\}^{**} = 0. \quad (70)$$

Eq. (68) reads that $j^+(z)$ has the properties of the Virasoro stress tensor for the single bosonic (Liouville) mode (51) of our model and is indeed the same as in the purely bosonic case [14] (this can be checked by use of Eqs. (11), (14) and taking into account the constraints):

$$T_m \equiv j^+ = k[(\partial\phi)^2 - \partial^2\phi]. \quad (71)$$

Thus in the classical case under consideration the full Virasoro stress tensor $T(z)$ and the superconformal current $G(z)$ constructed of $T_m(z)$, $b(z)$ and $c(z)$ have the following form:

$$\begin{aligned} T(z) &= T_m - \frac{3i}{2}b\partial c - \frac{i}{2}\partial bc, \\ G(z) &= ib + icT_m - bc\partial c - \frac{k}{4}c\partial c\partial^2 c - ik\partial^2 c. \end{aligned} \quad (72)$$

The $n = 1$ superconformal algebra of (72) realized on the Dirac brackets (66) is

$$\begin{aligned} \{T(z_1), T(z_2)\}^{**} &= \frac{k}{2}\partial^3\delta(z_1 - z_2) + \delta(z_1 - z_2)\partial T(z_2) - 2\partial\delta(z_1 - z_2)T(z_2), \\ \{T(z_1), G(z_2)\}^{**} &= \delta(z_1 - z_2)\partial T(z_2) - \frac{3}{2}\partial\delta(z_1 - z_2)T(z_2) \\ \{G(z_1), G(z_2)\}^{**} &= -2ki\partial^2\delta(z_1 - z_2) + 2\delta(z_1 - z_2)T(z_2). \end{aligned} \quad (73)$$

The central charge of the classical algebra is $c = 6k$.

The $n = 1$ superconformal system described by (72), (73) is a classical counterpart of the $b - c$ matter structure used in the framework of the universal string theory [2]. Hence, we can assert that in the case under consideration we deal with a particular example of an $n = (1, 1)$, $D = 3$ fermionic string in a physical gauge where two longitudinal bosonic modes are gauge fixed and the single (transversal) bosonic degree of freedom of the string is described by the Liouville mode (compare with the case of the $N = 2$, $D = 3$ GS superstring of Section 3). This gauge is alternative to the light-cone gauge of string theory, where also only the physical string modes remain.

We have thus established, at the classical level, links of the reduced supersymmetric $SL(2, \mathbf{R})$ WZNW model with the $N = 2$, $D = 3$ Green-Schwarz superstring on one hand and the $n = (1, 1)$, $D = 3$ fermionic string, carrying spin $(\frac{3}{2}, -\frac{1}{2})$ matter on the worldsheet, on the other hand. As to quantum relationship of the model to strings, the consideration of these points touches deep problems of comparing the results of quantization of constrained systems carried out before or after solving classical constraints and is beyond the scope of the present work.

So let us now turn to the quantization of the $n = 1$ superconformal algebra (72), (73) realized on the fields ϕ , b and c without addressing to string connection.

To quantize this algebra we should take into account operator ordering in (72) and pass from the Dirac brackets to the quantum (anti)commutation relations, or to consider

the quantum operator product expansion of the fields. We choose the second formalism, and as a result we get a realization of the quantum $n = 1$ superconformal algebra which arose in [47, 2].

In the OPE language the super-Virasoro algebra we reproduce is:

$$\begin{aligned}\underline{T(z)T(0)} &= \frac{c/2}{z^4} + \frac{2T}{z^2} + \frac{T'}{z}, \\ \underline{T(z)G(0)} &= \frac{(3/2)G}{z^2} + \frac{G'}{z}, \\ \underline{G(z)G(0)} &= \frac{2c/3}{z^3} + \frac{2T}{z}.\end{aligned}\tag{74}$$

Here and in the following formulas the fields on the r.h.s. are computed at $w = 0$. The prime denotes for simplicity the ∂_z derivative.

The OPE for the $b(z), c(z)$ field can be normalized as

$$\underline{c(z)b(0)} = \frac{1}{z},$$

and $j^+ \equiv T_m(z)$ satisfies the OPE for the bosonic matter stress tensor with the quantum Liouville central charge $c_m = 6k + \frac{k+4}{k-2}$ (see, for example, [46]) and has regular OPE's with $b(z)$ and $c(z)$ (compare with Eqs. (68)).

The realization of the $n = 1$ super-Virasoro algebra in terms of the fields $T_m(z)$, $b(z)$ and $c(z)$ was constructed in [2]:

$$\begin{aligned}T &= T_m - \frac{3}{2} : bc' : - \frac{1}{2} : b'c : + \frac{1}{2} \partial^2 (: cc' :), \\ G &= b+ : cT_m : + : bcc' : - \frac{c_m - 26}{24} : cc'c'' : + \frac{c_m - 11}{6} c''\end{aligned}\tag{75}$$

where dots denote the normal ordering, and the quantum central charge is $c = c_m - 11$. Notice the appearance of a new (the last) term in $T(z)$ and changing of coefficients of terms in $G(z)$ (75) in comparison with the classical case (72) due to the operator ordering.

To the $n = 1$ superconformal algebra (74) one can add the spin $(-\frac{1}{2})$ field $c(z)$ as the generator of the fermionic transformations (61). Then we get a nonlinear extension of the super-Virasoro algebra considered in [53].

For completeness let us present an alternative realization of the super-Virasoro algebra which makes use of a field $V(z)$ (conventionally represented as $V(z) \equiv \partial\Phi(z)$) satisfying the free-field OPE

$$\underline{V(z)V(0)} = -\frac{1}{z^2}\tag{76}$$

together with the free fields $b(z)$ and $c(z)$. It is given by

$$T = -\frac{1}{2} : V^2 : + qV' - \frac{3}{2} : bc' : - \frac{1}{2} : b'c :,$$

$$\begin{aligned}
G = & \frac{1}{2}b- : V^2c : - \frac{2}{q} : Vc' : - \frac{(1-2q^2)}{q} : V'c :, \\
& + 2 : cc'b : + (\frac{13}{3} - \frac{1}{q^2} - 4q^2) : cc'c'' : - \frac{2}{3}(5 - 6q^2)c'',
\end{aligned} \tag{77}$$

where $q = (\frac{c_m-1}{12})^{\frac{1}{2}}$, and the central charge is the same as above.

Notice that in this case the stress-energy tensor $T(z)$ is the same as in the classical case (72), i.e. the standard one given by the sum of two stress-energy tensors for the bosonic and fermionic free-field components, respectively.

6 Hamiltonian reduction in the case of a general superaffine Lie algebra

In this Section we shall discuss how to generalize the previous construction valid for the $sl(2)$ algebra to any given bosonic algebra (or even superalgebra). A complete analysis of this case is postponed to future work, here we restrict ourselves to outline the main features and basic ingredients.

To simplify discussion it is convenient to use the language of the so-called soldering procedure introduced by Polyakov in [48], which allows reproducing the results of the (more complete) Dirac analysis performed above.

Let us start recalling that the $n = 1$ chiral fermionic supercurrents $\Psi^\pm(Z)$, $\Psi^0(Z)$ of the superaffine $sl(2)$ transform under the infinitesimal gauge transformations parametrized by bosonic chiral superfields $\epsilon^\pm(Z)$, $\epsilon^0(Z)$ as follows:

$$\begin{aligned}
\delta\Psi^+ &= 2(\epsilon^0\Psi^+ - \epsilon^+\Psi^0) + \frac{1}{i}D\epsilon^+, \\
\delta\Psi^0 &= \epsilon^+\Psi^- - \epsilon^-\Psi^+ + \frac{1}{i}D\epsilon^0, \\
\delta\Psi^- &= 2(\epsilon^-\Psi^0 - \epsilon^0\Psi^-) + \frac{1}{i}D\epsilon^-.
\end{aligned} \tag{78}$$

Using the Maurer Cartan equation (10) we constructed from $\Psi^\pm(Z)$, $\Psi^0(Z)$ the bosonic superfields J^\pm , J^0 (11)

$$\begin{aligned}
J^+ &= D\Psi^+ - 2i\Psi^0\Psi^+, \\
J^0 &= D\Psi^0 + i\Psi^-\Psi^+, \\
J^- &= D\Psi^- + 2i\Psi^0\Psi^-.
\end{aligned} \tag{79}$$

By construction they obey the following $sl(2)$ transformation properties

$$\delta J^+ = 2(\epsilon^0 J^+ - \epsilon^+ J^0) + \partial\epsilon^+,$$

$$\begin{aligned}
\delta J^0 &= \epsilon^+ J^- - \epsilon^- J^+ + \partial \epsilon^0, \\
\delta J^- &= 2(\epsilon^- J^0 - \epsilon^0 J^-) + \partial \epsilon^-.
\end{aligned} \tag{80}$$

They correspond to a trivial supersymmetrization of the $sl(2)$ affine algebra obtained by replacing the ordinary currents and parameters with the chiral superfields J^i and ϵ^i . Since the ring structure properties of the superfields are the same as for the ordinary fields it is clear that any bosonic theory can be trivially supersymmetrized that way. In general it leads to uninteresting models. What makes things different here is the presence of the set of equations (79), which allow expressing the superfields J^i in terms of the basic superfields Ψ^i .

The constraint $J^- - 1 = 0$ and the gauge fixing condition $J^0|_{\theta=0} = 0$ can be consistently imposed as discussed in Section 2, 4 and 5. Then the only independent transformations which remain in (78) and (80) are parametrized by the bosonic component of $\epsilon^-(Z)$ which coincides with the conformal transformation parameter $\lambda(z)$ in (46)–(49), the fermionic component of $\epsilon^-(Z)$ which is connected with supersymmetry parameter $\epsilon(z)$ in (43)–(45) through the relation $\frac{1}{i}D\epsilon^-|_{\theta=0} = \epsilon(1 + i\psi^-\partial\psi^-) - \frac{1}{2}\partial\lambda\psi^-$, and an additional fermionic transformation parameter $D\epsilon^+|_{\theta=0}$ (compare with (61)).

This reasoning can be applied to get information about the reduction of more general $n = 1$ superaffine algebras.

We wish to stress that the bosonic superfields J^\pm associated to the roots of the $sl(2)$ algebra are obtained, as a consequence of the Maurer-Cartan equations (10), by applying to Ψ^\pm a fermionic derivative covariant with respect to the superaffine $U(1)$ subalgebra of $sl(2)$.⁶ This property obviously takes place in the case of a generic bosonic algebra. Let $\Psi^{\pm j}(Z)$ denote generic superfield currents associated to a root system of an $n = 1$ superaffinization of a given bosonic simple Lie algebra \mathcal{G} , and $\Psi^{0,\alpha}(Z)$ denote the supercurrents associated to the Cartan subalgebra (here $\alpha = 1, \dots, r$, and r is the rank of \mathcal{G}). The supercurrents $\Psi^{\pm j}(Z)$ associated to the simple and nonsimple roots are covariant w.r.t. the $n = 1$ superaffine subalgebra $\hat{U}(1)^r$ of \mathcal{G} , i.e. they satisfy the following Poisson brackets:

$$\{\Psi^{0,\alpha}(Z), \Psi^{\pm j}(W)\} = \pm q^{\alpha,j} \Psi^{\pm j}(W) \delta(Z - W).$$

where $q^{\alpha,\pm j} = \pm q^{\alpha,j}$ is a set of $U(1)$ -charges. When $+j = \beta$ labels a simple root, $q^{\alpha,\beta}$ coincides with the Cartan matrix $K_{\alpha\beta}$.

⁶Covariant derivatives to analyze algebra structures have been introduced in [49] and further employed in [50] in connection with integrable hierarchies; fermionic covariant derivatives have been introduced in [51].

The bosonic superfields $J^{\pm j}$ associated to the roots are constructed from Ψ as

$$J^{\pm j} = \mathcal{D}\Psi^{\pm j} + i \sum_{k,l} c^{\pm j}_{k,l} \Psi^k \Psi^l, \quad (81)$$

where the covariant derivative is uniquely determined as $\mathcal{D} \equiv D + \sum_{\alpha=1,\dots,r} q^{\alpha,j} \Psi^{0,\alpha}$ by the requirement that $J^{\pm j}$ satisfy the Maurer–Cartan equations and are covariant with the charge $q^{\alpha,j}$:

$$\{\Psi^{0,\alpha}(Z), J^{\pm j}(W)\} = \pm q^{\alpha,j} J^{\pm j}(W) \delta(Z - W)$$

The constants $c^{\pm j}_{k,l}$ in (81) are the Lie algebra structure constants; they are non-vanishing only for those integral values of k, l for which $q^{\alpha,k} + q^{\alpha,l} = \pm q^{\alpha,j}$ for any α .

Similarly the bosonic supercurrents $J^{0,\alpha}$ associated to the Cartan subalgebra are given by

$$J^{0,\alpha} = D\Psi^{0,\alpha} + i \sum_j c^{\alpha}_{+j,-j} \Psi^{+j} \Psi^{-j}$$

(here again the coefficients $c^{\alpha}_{+j,-j}$ are Lie algebra structure constants).

Now the set of constraints for the so-called abelian reduction can be consistently imposed as in the supersymmetric $sl(2)$ and purely bosonic case: the set of superfields $J^{-1,i}$ (81) associated to the simple (negative) roots are put equal to one: $J^{-\alpha} = 1$, while the remaining generators associated to lower (negative) roots are set equal to 0, as well as the bosonic components of the Cartan supercurrents $J^{0,\alpha}$. The soldering procedure and the Dirac analysis for these constraints can be made as in the $sl(2, \mathbf{R})$ case. As a result superaffine transformations are converted into extended super–Virasoro transformations and one may get an $n = 1$ supersymmetric extension of the bosonic \mathcal{W} algebra realized on the bosonic \mathcal{W} algebra generators and r pairs of free fermions.

It deserves mentioning that the standard supersymmetrization of a bosonic \mathcal{W} algebra is carried out in two steps (see [52]), namely, by identifying a suitable superalgebra and an embedding of $osp(1|2)$ or $sl(1|2)$ into it. In our approach we perform the supersymmetrization of the \mathcal{W} algebra by direct use of the same ingredients as in the bosonic case: the classical Lie algebra as a base and a given embedding of $sl(2)$ into the former (the principal embedding for abelian theories). Because of the presence of spin $(\frac{3}{2}, -\frac{1}{2})$ $b - c$ systems such a realization of a super- \mathcal{W} algebra differs in field contents from the standard ones based on superalgebras described by the simple fermionic root system [52]. A reduction procedure alternative to ours was used in [47] to get $n = 2$ superconformal $b - c$ structures and realizations of super- W_n algebras from the $Sl(n|n-1)$ superalgebras in a Gauss decomposition containing simple bosonic roots. A class of supersymmetric \mathcal{W}

algebras realized with the use of fermionic $b - c$ systems has also been considered in [53] and references therein.

Super-Toda equations obtained with the procedure under consideration are a generalization of Eqs. (15), (16):

$$\bar{D}D\Phi_\alpha = e^{\sum_\beta K_{\alpha\beta}\Phi_\beta}\bar{\Psi}^{+\alpha}\Psi^{-\alpha}, \quad \bar{D}\Psi^{-\alpha} = 0 = D\bar{\Psi}^{+\alpha}; \quad (82)$$

$$D\Psi^{-\alpha} + \sum_\beta K_{\alpha\beta}\Psi^{0,\beta}\Psi^{-\alpha} = 1, \quad \bar{D}\bar{\Psi}^{+\alpha} + \sum_\beta K_{\alpha\beta}\bar{\Psi}^{0,\beta}\bar{\Psi}^{+\alpha} = 1. \quad (83)$$

In Eqs. (82), (83) there is no summation over α ; $\Phi_\alpha(Z, \bar{Z})$ are the Cartan subalgebra superfields of the Gauss decomposition of the group element

$$G(Z, \bar{Z}) = e^{\sum \beta^\alpha E_{+\alpha}} e^{\sum \Phi_\alpha H_\alpha} e^{\sum \gamma^\alpha E_{-\alpha}}$$

and (with taking into account the constraints imposed)

$$\Psi^{0,\alpha} = \frac{1}{i}D\Phi_\alpha + \beta^\alpha\Psi^{-\alpha}, \quad \bar{\Psi}^{0,\alpha} = i\bar{D}\Phi_\alpha + \gamma^\alpha\bar{\Psi}^{-\alpha} \quad (\text{no summation over } \alpha).$$

Notice that because of the form of $\Psi^{0,\alpha}$, $\bar{\Psi}^{0,\alpha}$ and the summation over β in (83) the constraints (83) do not reduce to relations which contain only $D\Phi_\alpha$ (in contrast to Eqs. (12), (16)). A consequence of this is that, unlike in the simplest $Sl(2, \mathbf{R})$ case, fermions do not decouple from bosons in the component form of Eqs. (82). (At least we have not managed to find a local field redefinition which would allow one to completely decouple fermions).

The detailed studying of super-Toda models of this kind may be of interest because of unusual supersymmetric properties, their connection with standard super-Toda models, and of possible physical applications.

The Hamiltonian reduction procedure outlined herein can also be applied to those superalgebras which always contain in their root decomposition bosonic simple roots. In this case one should impose a “mixed” type of constraints, namely, standard ones associated to fermionic simple roots and nonlinear ones for the bosonic roots as discussed in this paper. It is very likely that the models derived this way have a nontrivial interaction between the bosonic and fermionic sector. As far as we know, without explicit breaking supersymmetry [18], superalgebras of this kind have not been involved yet into the production of super-Toda models. The complete classification of superalgebras and their Dynkin diagrams can be found in [54]. From this paper we learn that, for instance, up to rank 3, there exist 6 superalgebras which contain bosonic simple roots in any root decomposition. The rank 2 superalgebra of this kind is $Osp(1|4)$, and the rank 3 superalgebras are $Osp(5|2)$, $Osp(2|4)$, $Osp(1|6)$, $Sl(1|3)$ and $G(3)$.

Apart from these systems which deserve to be analyzed in detail, there is another problem which is worth studying. Many superalgebras admit different variants of root decomposition. Even superalgebras which are expressible in terms of only fermionic simple roots can be expressed, in another basis, in terms of a Dynkin diagram involving bosonic simple roots (the simplest superalgebra having this property is $Sl(1|2)$). It is interesting to understand possible relations, if any, between super-Toda models derived from different presentations of the same superalgebra.

7 Conclusion and discussion

We have studied the Hamiltonian reduction of the $n = (1, 1)$ supersymmetric WZNW models having a classical (bosonic) group as target space.

Since the simple roots of the corresponding Lie algebra are bosonic the constraints are to be imposed on associated bosonic supercurrents constructed out of the basic fermionic supercurrents in a nonlinear way prescribed by the Maurer–Cartan equation. The constraints thus obtained are a mixture of bosonic first-class and fermionic second-class constraints. This makes difference between the Hamiltonian reduction of the models considered above and the conventional Hamiltonian reduction of the bosonic WZNW models [14, 15] and the supersymmetric WZNW models based on superalgebras admitting realization in terms of fermionic simple roots only [17]. In the latter case all the constraints imposed are of the first class.

Hamiltonian reduction results in shifting the conformal spins of bosonic as well as of fermionic current fields. As a result we obtained the fermionic $b-c$ system with spin $\frac{3}{2}$ and $-\frac{1}{2}$. Supersymmetry transforming these fields is realized nonlinearly and is spontaneously broken.

The supersymmetric Toda systems obtained by reducing the WZNW models with bosonic groups contain a free left–right–chiral fermion sector. In the simplest case of the $Sl(2, \mathbf{R})$ group the fermionic sector completely decouples from the bosonic Liouville equation. While in the case of super-Toda models based on higher rank bosonic groups fermions contribute to the r.h.s. of the bosonic equations. Thus the supersymmetric generalization of the bosonic Toda-models obtained this way is alternative to the conventional one (see e.g.[55]) which involves nontrivial interactions in both the bosonic and fermionic sector and where all fermionic fields have positive conformal spin. It seems of interest to analyse in detail the relationship between these two supersymmetric versions.

An important problem which has remained unsolved yet is the construction of a gauged WZNW action from which one can directly get the constraints as equations of motion

of auxiliary gauge fields. As we have already mentioned in the main text, the problem is caused by the nonlinear form of the constraints which, in addition, contain bosonic coordinate derivatives of group-valued superfields $G(Z)$, while the WZNW action (1) is constructed with the use of only Grassmann supercovariant derivatives. The cause of the problem seems akin to the problem of constructing superfield actions for n -extended WZNW models.

Possible directions of generalizing results obtained are following.

The detailed analysis of the Hamiltonian reduction of the $Sl(2, \mathbf{R})$ WZNW model at the classical level revealed its connection with $N = 2$ Green–Schwarz superstrings and $n = (1, 1)$ fermionic strings propagating in flat $D = 3$ space–time. The extension to the quantum level and the analysis of links of the present model with the standard $n = (1, 1)$, $D=3$ Neveu–Schwarz–Ramond string whose fermionic matter fields have conformal spin $\frac{1}{2}$ require additional study since it involves nontrivial field redefinitions when passing from one superstring formulation to another (see, for example [22, 23, 24]).

Studying higher-dimensional target spaces seems also of interest. As we know from the doubly supersymmetric approach (see [4] and references therein), in $D=4$ flat superspace–time, whose structure group is $Sl(2, \mathbf{C})$, Green–Schwarz superstrings are described by an $n = 2$ worldsheet supersymmetry. Hence we can associate to these strings an $n = 2$ supersymmetric $Sl(2, \mathbf{C})$ WZNW model [27] appropriately reduced the way considered above. For the $N = 2$, $D = 4$ superstring this should result in a supersymmetric integrable system ⁷ which describes the physical modes of the classical superstring and which consists of a bosonic Liouville system and two pairs of left–right–moving free fermions. As in the $D=3$ case, one may also expect a connection with an $n = (2, 2)$ superconformal system corresponding to an $n = (2, 2)$ fermionic string [2].

Next, more complicated, step is to perform the Hamiltonian reduction of supersymmetric WZNW models based on $SO(1, 5)$ and $SO(1, 9)$ group which correspond, respectively to $N = 2$ Green–Schwarz superstrings in $D = 6$ and $D = 10$ space–time with $n = (4, 4)$ and $n = (8, 8)$ supersymmetries on the worldsheet, or at least with $n = (2, 2)$ manifest worldsheet supersymmetries, and to establish a connection with results of Refs. [2, 23, 24] on Neveu–Schwarz–Ramond and Green–Schwarz superstrings. Another words one should study a generalization of the Hamiltonian reduction procedure considered here to the case of WZNW models with extended supersymmetry.

Though we have talked of $N = 2$, Green–Schwarz superstrings in $D=3,4,6$ and 10 dimensions and corresponding $n = (D - 2, D - 2)$ superconformal models the discussion

⁷A relation of four–dimensional $n=2,4$ supersymmetric string backgrounds to integrable models has been already considered, for instance, in [56]

above is also valid for $N = 1$ Green–Schwarz superstrings and chiral (heterotic) $n = (D - 2, 0)$ superconformal models.

Another direction of study, which may turn out the most interesting from a mathematical physics point of view, is to apply the Hamiltonian reduction procedure to constrain supersymmetric WZNW models based on superalgebras with fermionic and bosonic simple roots in any Dynkin diagram [54]. The simplest example is the $OSp(1|4)$ superalgebra whose root decomposition contains one bosonic and one fermionic simple root. A potentially promising investigation concerns other supergroups of this kind, the three–rank exceptional supergroup $G(3)$ and the four–rank exceptional supergroup $F(4)$. It is likely [57] that their reductions are related respectively to an $n = 7$ and $n = 8$ supersymmetric extension of the conformal algebra.

Acknowledgements.

The authors are grateful to I. Bandos, F. Bastianelli, E. Ivanov, S. Ketov, S. Krivonos, A. Pashnev, P. Pasti, A. Sorin, M. Tonin and D. Volkov for interest to this work and valuable discussion.

References

- [1] I. Bandos, D. Sorokin and D. Volkov, *Phys. Lett.* **B372** (1996) 77 (hep–th/9510220).
- [2] N. Berkovits and C. Vafa, *Mod. Phys. Lett.* **A 9** (1993) 653.
N. Berkovits and N. Ohta, *Phys. Lett.* **B 334** (1994) 72.
F. Bastianelli, *Phys. Lett.* **B 322** (1994) 340.
F. Bastianelli, N. Ohta and J. L. Peterson, *Phys. Lett.* **B 327** (1994) 35.
N. Berkovits and C. Vafa, *Nucl. Phys.* **B 433** (1995) 123.
- [3] A. Galperin and E. Sokatchev, *Phys. Rev.* **D48** (1993) 4810.
- [4] I. Bandos, P. Pasti, D. Sorokin, M. Tonin and D. Volkov, *Nucl. Phys.* **B446** (1995) 79.
- [5] I. Bandos, D. Sorokin and D. Volkov, *Phys. Lett.* **B 352** (1995) 269.
- [6] M. Chaichian and P. P. Kulish, *Phys. Lett.* **B 78** (1978) 413.
- [7] A. M. Polyakov, *Phys. Lett.* **B 103** (1981) 210.

- [8] S. P. Novikov, *Sov. Math. Dokl.* **37** 1982 3.
E. Witten *Commun. Math. Phys.* **92** (1984) 83.
- [9] P. Di Vecchia, V. G. Knizhnik, J. L. Petersen and P. Rossi, *Nucl. Phys.* **B 253** (1985) 701.
- [10] E. Abdalla and M. C. B. Abdalla, *Phys. Lett.* **B 152** (1985) 59.
- [11] P. Goddard, A. Kent and D. Olive, *Phys. Lett.* **B 152** (1985) 88.
- [12] Y. Kazama and H. Suzuki, *Phys. Lett.* **B 216** (1989) 112; *Nucl. Phys.* **B 321** (1989) 232.
- [13] M.B. Halpern, E. Kiritsis, N.A. Obers and K. Clubok, *Phys. Reports* **265** (1996) 1.
- [14] P. Forgács, A. Wipf, J. Balog, L. Fehér and L. O’Raifeartaigh, *Phys. Lett.* **B 227** (1989) 214.
J. Balog, L. Fehér, L. O’Raifeartaigh, P. Forgács and A. Wipf, *Ann. of Phys.* **203** (1990) 76.
- [15] L. Fehér, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, *Phys. Reports* **222** (1992) 1.
- [16] T. Inami and K. I. Izawa, *Phys. Lett.* **B 255** (1991) 521.
- [17] F. Delduc, E. Ragoucy and P. Sorba, *Commun. Math. Phys.* **146** (1992) 403.
- [18] M. A. Olshanetsky, *Commun. Math. Phys.* **88** (1983) 63.
J. Evans and T. Hollowood, *Nucl. Phys.* **B 352** (1991) 723.
- [19] A. N. Redlich and H. Schnitzer, *Phys. Lett.* **B 167** (1986) 315.
D. Gepner and E. Witten, *Nucl. Phys.* **B 278** (1986) 493.
J. Balog, L. O’Raighfertaigh, P. P. Forgács and A. Wipf, *Nucl. Phys.* **B 325** (1989) 225.
I. Bars and D. Nemeshansky, *Nucl. Phys.* **B 348** ((1991)) 89.
I. Bars, Preprint USC-95/HEP-B3, hep-th/9511187; *Phys. Rev.* **D 53** (1996) 3308.
- [20] E. Witten, *Phys. Rev.* **D 44** (1991) 314.
I. Bars and K. Sfetsos, *Mod. Phys. Lett.* **A 7** (1992) 1091; *Phys. Rev.* **D 46** (1992) 4495.
P. Ginsparg and F. Quevedo, *Nucl. Phys.* **B 385** (1992) 527.
C. Kounnas and D. Lüst, *Phys. Lett.* **B 289** (1992) 56.

- A. Giveon, E. Rabinovici and A. A. Tseytlin, *Nucl. Phys.* **B 409** (1993) 339.
K. Sfetsos and A. A. Tseytlin, *Nucl. Phys.* **B 415** (1994) 116.
C. Klimčík and A. A. Tseytlin, *Nucl. Phys.* **B 424** (1994) 71.
E. Ragoucy, A. Sevrin and P. Sorba, Preprint ENSLAPP-555/95, VUB-TH.395, hep-th/9511049.
- [21] M. Green, J. Schwarz and E. Witten, *Superstring Theory*, CUP, 1987 (and references therein).
- [22] D. V. Volkov and A. Zheltukhin, *Sov. Phys. JETP Lett.* **48** (1988) 61; *Letters in Math. Phys.* **17** (1989) 141; *Nucl. Phys.* **B 335** (1990) 723.
D. Sorokin, V. Tkach, D. V. Volkov and A. Zheltukhin, *Phys. Lett.* **B216** (1989) 302.
S. Aoyama, P. Pasti and M. Tonin, *Phys. Lett.* **B283** (1992) 213.
- [23] N. Berkovits, *Nucl. Phys.* **B 420** (1994) 332, *Nucl. Phys.* **B 431** (1994) 258.
- [24] N. Berkovits, Preprint IFUSP-P-1212, April 1996, hep-th/9604123.
- [25] K. Gawedzki and A. Kupianen, *Phys. Lett.* **B 215** (1988) 19; *Nucl. Phys.* **B 320** (1989) 625.
P. Bowcock, *Nucl. Phys.* **B 316** (1989) 80.
- [26] A. N. Leznov and M. V. Saveliev, *Letters in Math. Phys.* **3** (179) 489; *Commun. Math. Phys.* **74** (1980) 111.
- [27] C. M. Hull and B. Spence, *Phys. Lett.* **B 241** (1990) 357; *Nucl. Phys.* **B 345** (1990) 493.
- [28] C. Ahn, E. Ivanov and A. Sorin, Preprint JINR E2-95-343, July, 1995, hep-th/9508005.
- [29] L. Eisenhart, “Riemannian Geometry”, Princeton University Press, Princeton, NJ, 1964.
- [30] F. Lund and T. Regge, *Phys. Rev.* **D 14** (1976) 1524.
- [31] R. Omnès, *Nucl. Phys.* **B 149** (1979) 269.
- [32] B. M. Barbashov and V. V. Nesterenko, *Introduction to the relativistic string theory*. World Scientific, Singapore, 1990 (and references therein).
- [33] A. L. Larsen and N. Sánchez, *Phys. Rev.* **D 54** (1996) 2801.

- [34] I. Bakas and K. Sfetsos, *Phys. Rev. D* **54** (1996) 3995.
- [35] V. I. Ogievetskii and E. S. Sokatchev, *Phys. Lett. B* **79** (1978) 222; *Sov. J. Nucl. Phys. (Yadern.Fiz.)* **28** (1978) 1631, *Ibid.* **31** (1980) 264, 821.
A. S. Schwarz, *Commun. Math. Phys.* **87** (1982) 37; A. A. Roslyi and A. S. Schwarz, *Commun. Math. Phys.* **95** (1984) 161; *Commun. Math. Phys.* **96** (1984) 285.
- [36] J. De Azcarraga and J. Lukiersky, *Phys. Lett. B* **113** (1982) 170.
W. Siegel, *Phys. Lett. B* **128** (1983) 397.
- [37] D. Sorokin, V. Tkach and D. Volkov, *Mod. Phys. Lett. A* **4** (1989) 901.
- [38] S. J. Gates Jr. and H. Nishino, *Class. Quantum Grav.* **3** (1986) 391. J. Kowalski-Glikman, *Phys. Lett. B* **180** (1986) 358. R. Brooks, F. Muhammed and S. J. Gates Jr. *Class. Quantum Grav.* **3** (1986) 745. J. Kowalski-Glikman, J. W. van Holten, S. Aoyama and J. Lukierski, *Phys. Lett. B* **201** (1987) 487. A. Kavalov and R. L. Mkrtchyan, Preprint Yer.PhI 1068(31)-88, Yerevan, 1988 (unpublished).
- [39] P. S. Howe, E. Sezgin, Superbranes. Preprint CERN-TH-96-200, Jul 1996. 12pp. hep-th/9607227
- [40] I. Bandos, hep-th/9510213.
- [41] P. S. Howe, *J. Phys. A* **12** (1979) 393; M. Brown and S. J. Gates Jr., *Ann. of Phys.* **122** (1979) 443.
- [42] D. V. Volkov and V. P. Akulov, *Sov. Phys. JETP Lett.* **16** (1972) 438, *Phys. Lett. B* **46** (1973) 109.
D. V. Volkov and V. A. Soroka, *Sov. Phys. JETP Lett.* **18** (1973) 312.
- [43] E. Ivanov, S. Krivonos and A. Pashnev, private communication.
- [44] K. Sundermeyer, “Constrained Dynamics” Lecture Notes in Physics, Vol. 169, Springer-Verlag, Berlin, 1982.
- [45] V. Drinfeld and V. Sokolov, *J. Sov. Math.* **30** (1984) 1975.
- [46] N. Ishibashi, *Nucl. Phys. B* **379** (1992) 199.
- [47] M. Bershadsky, W. Lerche, D. Nemeschansky and N. P. Warner, *Nucl. Phys. B* **401** (1993) 304.
- [48] A. M. Polyakov, *Int. J. Mod. Phys. A* **5** (1990) 833.

- [49] F. Delduc, L. Frappat, E. Ragoucy, P. Sorba and F. Toppan, *Phys. Lett.* **B 318** (1993) 457.
- [50] F. Toppan, *Phys. Lett.* **B 327** (1994) 249.
- [51] F. Toppan, *Int. J. Mod. Phys.* **A 10** (1995) 895.
- [52] L. Frappat, E. Ragoucy and P. Sorba, *Comm. Math. Phys.* **157** (1993) 499.
- [53] S. Krivonos and A. Sorin, preprint JINR E2-95-427, hep-th 9510072.
- [54] L. Frappat, A. Sciarrino and P. Sorba, *Comm. Math. Phys.* **121** (1989) 457.
- [55] F. Toppan, *Phys. Lett.* **B 260** (1991) 346;
F. Toppan and Y.-Z. Zhang, *Phys. Lett.* **B 292** (1992) 67.
- [56] G. L. Cardoso and D. Lüst, *Phys. Lett.* **B 345** (1995) 220.
- [57] M. Günaydin and S. Ketov, *Nucl. Phys.* **B 467** (1996) 215.